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THE STABILITY OF THE EQUILIBRIUM POSITION OF SCLERONOMOUS MECHANICAL SYSTEMS†

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The problem of the stability of the equilibrium position of a scleronomic mechanical system is considered. The comparison method enables this problem to be reduced to the problem of the stability of scalar differential equations. The stability conditions are found for certain types of scalar comparison equations (Sections 1–4), and the sufficient conditions for the stability of the equilibrium positions of various scleronomous mechanical systems are determined from these (Sections 5–9).

1. INTRODUCTION

Consider the scalar differential equation

$$\dot{u} = a(t)f(u) + G(t,u) \tag{1.1}$$

where $a : R^+ \to R, f : (0, U) \to (0, \infty)$ $(0 < U \leq +\infty)$ and $G : R^+ \times (0, U) \to R^+$ are continuous functions satisfying the conditions

$$\int_0^t a(s)ds \leq a_*, \quad \forall t \in R^+, \quad a_* > 0 \tag{1.2}$$

$$\int_0^u \frac{ds}{f(s)} = \infty, \quad \forall u \in (0, U) \tag{1.3}$$

These conditions guarantee the stability of the solution $u = 0$ of the unperturbed equation $u = a(t)f(u)$ (see, e.g. [1]).

In the first part of this paper (Sections 2–4), the problem of the stability of the unperturbed solution $u = 0$ for various assumptions with regard to the perturbation $G(t, u)$ is analysed. In Section 2 the case

$$G(t,u) = b(t)f(u)\varphi(AF) \tag{1.4}$$

is considered, where $b : R^+ \to R^+$ $(b(t) \not\equiv 0)$ and $\varphi : (0, \infty) \to (0, \infty)$ are continuous functions, and the functions $A(t)$ and $F(u)$ are defined by the equations [1, 2]

$$A(t) = \exp\left(-\int_0^t a(s)ds\right), \quad F(u) = \exp\left(\int_{\bar{u}}^u \frac{ds}{f(s)}\right) \tag{1.5}$$

for a certain $\bar{u} \in (0, U)$.

Remark 1.1. It follows from conditions (1.2) and (1.3) that $A(t) \geq \exp(-a_*)$ in R^+, the function $F(u)$ is defined and strictly increases in $(0, U)$, $F(u) \to 0$ as $u \to 0$. Hence, the function $F(u)$ can be considered as a function of the Hahn type. As is well known [3], the function $\gamma : R^+ \to R^+$, which increases strictly monotonically with a value $\gamma(0) = 0$, is of this type. We will henceforth denote the class of such functions by K.

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In the case when $\varphi(s) = s^k$ (k = const $\neq 0$), Eq. (1.1) can be converted to the form

$$\dot{u} = a(t)\frac{F(u)}{F'(u)} + \beta(t)\frac{F(u)^{k+1}}{F'(u)}, \quad \beta(t) = b(t)A^k(t), \quad F'(u) = \frac{dF}{du}$$

This equation is called the *"extended" Bernoulli equation* (see, e.g. [4]) and when $f(u) = u$ is identical with the classical Bernoulli equation.

Theorems of the stability of the solution $u = 0$ are obtained in Section 3 for the case when $\varphi(s) = s^k$ in Eq. (1.4).

In Section 4 it is assumed that $a(t) \geqslant 0$ in R^+ and

$$G(t,u) = \beta(t)g(u)$$

where $g : (0, U) \to (0, \infty)$ is a continuous function.

Remark 1.2. Suppose $t_0 \in R^+$ and $u_0 \in (0, U)$ are given. We will denote the solution of the Cauchy problem

$$\dot{u} = a(t)f(u) + G(t,u), \quad u(t_0) = u_0$$

by $u(t) = u(t, t_0, u_0)$ and its maximum interval of existence by $I = [t_0, \omega)$, $\omega \leqslant \infty$. Using the comparison method it can be proved that in the cases considered

$$u(t,t_0,u_0) > 0, \quad \forall t \in I$$

and, consequently, if $\omega < \infty$, then $u(t, t_0, u_0) \to U$ as $t \to \omega^-$.

In what follows, without citing the well-known definitions of stability, we will use the following definitions.

Definition 1.1. Solution $u = 0$ is called an *equi-attractive* solution, if $\forall t_0 \geqslant 0$, $\exists \delta \in (0, U)$, $\forall \varepsilon \in (0, U)$, $\exists T > 0$, $\forall u_0 \in (0, \delta)$ $u(t, t_0, u_0) < \varepsilon$, $\forall t \geqslant t_0 + T$.

Definition 1.2. Solution $u = 0$ is called an *equi-asymptotically stable* solution, if it is stable and equi-attractive.

Definition 1.3. The unperturbed solution $u = 0$ is called an *eventually stable* solution, if $\forall \varepsilon \in (0, U)$, $\exists T > 0$, $\forall t_0 \geqslant T$, $\exists \delta \in (0, U)$, $\forall u_0 \in (0, \delta)$ $u(t, t_0, u_0) < \varepsilon$, $\forall t \geqslant t_0$.

Definition 1.4. Solution $u = 0$ is called an *eventually uniformly stable* solution, if the constant δ in Definition 1.3 does not depend on t_0.

In the second part of the paper (Sections 5–9), the stability of the equilibrium position of a mechanical system with holonomic time-independent constraints under the action of explicitly time-dependent forces is considered. Problems of stability are examined for cases when the potential energy is not positive definite and even not non-negative for a fixed time.

In Section 5 the formulation of the problem is given. In Section 6 differential comparison equations are formulated. In Section 7 the conditions of stability of the zero equilibrium position of a mechanical system with respect to velocities are derived, and in Section 8 they are derived with respect to velocities and coordinates. In Section 9 specific examples are considered.

2. THE CASE OF A PERTURBATION $G(t, u) = b(t)f(u)\varphi(AF)$

We will consider the equation

$$\dot{u} = a(t)f(u) + b(t)f(u)\varphi(AF) \tag{2.1}$$

depending on the convergence or divergence of the integral

$$\int_0^r \frac{ds}{s\varphi(s)}, \quad r > 0 \tag{2.2}$$

We define the function

$$\Phi(r)=\int_0^r \frac{ds}{s\varphi(s)} \quad \text{or} \quad \Phi(r)=\int_c^r \frac{ds}{s\varphi(s)} \tag{2.3}$$

for certain $c > 0$ corresponding to these assumptions ($c = +\infty$ is also possible). In both cases $\Phi(r)$ is a strictly monotonically increasing function, defined in the interval $(0, \infty)$ with the set of values $(0, l)$ and $(-\infty, l)$ $(l > 0)$, respectively.

Lemma 2.1. Suppose $t_0 \in R^+$ and $u_0 \in (0, U)$ are given. Then

$$A(t)F(u(t)) = \Phi^{-1}(\Phi(v_0) + B(t_0,t)), \quad \forall t \in I \tag{2.4}$$

$$v_0 = A(t_0)F(u_0), \quad B(t_0,t) = \int_{t_0}^{t} b(s)ds$$

where Φ^{-1} is the inverse function of Φ and $u(t) = u(t, t_0, u_0)$ is the solution of Eq. (2.1).

Proof. We will assume that $v(t) = A(t)F(u(t))$ for any $t \in I$; then from Eq. (2.1) we obtain the following equality

$$\dot{v}(t) = b(t)v(t)\varphi(v(t)), \quad \forall t \in I$$

Hence and from Remark 1.1 it follows that $v(t) > 0$ in I and this also means that $v(t)\varphi(v(t)) > 0$ in I.

Further by separating the variables and integrating, we obtain

$$\Phi(v(t)) - \Phi(v_0) = B(t_0,t)$$

Below we will denote the class of continuous functions $b : R^+ \to R^+$ by $L^1(R^+)$,

$$\int_0^\infty b(t)dt < +\infty$$

Theorem 2.1. If integral (2.2) is diverging and the function $b(t) \in L^1(R^+)$, then the solution $u = 0$ is stable.

Given the additional assumption

$$A(t) \leqslant A_1 = \text{const}, \quad \forall t \in R^+$$

this stability is uniform. If $A(t) \to \infty$ as $t \to \infty$, then the solution $u = 0$ is equi-asymptotically stable.

Proof. We will fix $t_0 \geqslant 0$ and $\varepsilon \in (0, U)$ and assume that $\delta = \delta(t_0, \varepsilon)$ is a constant satisfying the relation

$$A(t_0)F(\delta) = \Phi^{-1}(\Phi(e^{-a_*}F(\varepsilon)) - B(t_0,\infty))$$

(the constant a_* is defined by condition (1.2)). Consequently, we obtain

$$\Phi(A(t_0)F(\delta)) + B(t_0,t) \leqslant \Phi(e^{-a_*}F(\varepsilon)), \quad \forall t \geqslant t_0 \tag{2.5}$$

It is obvious that $F(\delta) < F(\varepsilon)$; consequently, $0 < \delta < \varepsilon < U$. If the point $u_0 \in (0, \delta)$ is fixed arbitrarily, the function $F(u)$ will be defined at the point u_0 and $A(t_0)F(u_0) < A(t_0)F(\delta)$. From Eqs (2.4) and (2.5) we obtain the inequality

$$\Phi(A(t)F(u(t))) < \Phi(e^{-a_*}F(\varepsilon)), \quad \forall t \in I$$

But $\Phi(r)$ and $F(u)$ are strictly increasing functions and $A(t) \geqslant e^{-a_*}$ in I. This implies that $u(t) < \varepsilon$ in I and $I = [t_0, \infty)$, and it means that the solution $u = 0$ is stable.

For condition $A(t) \leqslant A_1$ = const the number $\delta(\varepsilon)$, $0 < \delta < \varepsilon$, determined from the equation

$$A_1 F(\delta) = \Phi^{-1}(\Phi(e^{-a_*} F(\varepsilon)) - B(0,\infty))$$

does not depend explicitly on t_0. Consequently, the stability of the solution $u = 0$ will be uniform.

We will show that if $A(t) \to +\infty$ for $t \to +\infty$, then the solution $u = 0$ of the unperturbed equation will be equi-attractive. For this condition for $\delta \in (0, U)$, $t_0 \in R^+$ and $\varepsilon \in (0, U)$ a number $T = T(t_0, \varepsilon) > 0$ can be found such that

$$A(t)F(\varepsilon) \geqslant \Phi^{-1}(\Phi(A(t_0)F(\delta)) - B(t_0,\infty)), \quad \forall t \geqslant t_0 + T \tag{2.6}$$

We will fix the arbitrary number $u_0 \in (0, U)$. Since

$$\Phi(A(t_0)F(u_0)) + B(t_0,t) < \Phi(A(t_0)F(\delta)) + B(t_0,\infty), \quad \forall t \geqslant t_0$$

then, comparing Eqs (2.4) and (2.6) it is easy to obtain that $A(t)F(u(t)) < A(t)F(\varepsilon)$, $\forall t \geqslant t_0 + T$ and, consequently, $u(t) < \varepsilon$, $\forall t \geqslant t_0 + T$. This concludes the proof.

Theorem 2.2. The solution $u = 0$ of Eq. (2.1) is unstable for condition (2.2).

Proof. We will fix the number $t_0 \in R^+$, so that $b(t_0) > 0$. Let $\varepsilon \in (0, U)$ be a constant, satisfying the inequality

$$F(\varepsilon) < \sup_{t \geqslant t_0} \left\{ \frac{1}{A(t)} \Phi^{-1}(B(t_0,t)) \right\} \tag{2.7}$$

Using the method of proof by contradiction, we will suppose that a $u_0 \in (0, U)$ exists such that $u(t) = u(t, t_0, u_0) < \varepsilon$ at $[t_0, \infty)$. Then, using Eq. (2.4) we obtain

$$\frac{1}{A(t)} \Phi^{-1}(\Phi(A(t_0)F(u_0)) + B(t_0,t)) < F(\varepsilon), \quad \forall t \geqslant t_0$$

which contradicts inequality (2.7).

The following theorem demonstrates that instability does not mean the loss of eventual stability.

Theorem 2.3. If condition (2.2) and also the condition

$$\Phi^{-1}(B(t_0,t))/(A(t)) \to 0 \quad \text{as} \quad t \to \infty$$

are satisfied, then the solution $u = 0$ of the unperturbed equation is eventually stable.

Proof. From the conditions of the theorem it follows that a constant $T = T(\varepsilon)$ exists such that

$$0 < \Phi^{-1}(B(0,t)) < A(t)F(\varepsilon), \quad \forall t \geqslant T \tag{2.8}$$

We will use $t_0 \geqslant T$ and denote by $\delta = \delta(t_0, \varepsilon) < \varepsilon$ the number satisfying the relation

$$A(t_0)F(\delta) = \Phi^{-1}(B(0,t_0))$$

We will fix u_0 from the interval $(0, \delta)$. Since $F(u_0) < F(\delta)$, it follows that

$$\Phi(A(t_0)F(u_0)) + B(t_0,t) < B(0,t_0), \quad \forall t \geqslant t_0$$

Hence, using Eqs (2.4) and (2.8), we obtain

$$A(t)F(u(t)) < A(t)F(\varepsilon), \quad \forall t \geqslant t_0$$

Consequently $u(t) < \varepsilon$ in $[t_0, \infty)$.

The following theorem can be proved similarly.

Theorem 2.4. If inequality (2.2) holds and also
(1) $b(t) \in L^1(R^+)$;
(2) a constant $A_1 < \infty$ exists such that $A(t) \leqslant A_1$ in R^+;
then the solution $u = 0$ is eventually uniformly stable.

3. THE CASE OF A PERTURBATION $G(t, u) = \beta(t)f(u)F^k(u)$

Consider the scalar differential equation obtained from (2.1) for $\varphi(s) = s^k$,

$$\dot{u} = a(t)f(u) + \beta(t)g(u), \quad g(u) = f(u)\exp\left\{k\int_{\bar{u}}^{u} \frac{ds}{f(s)}\right\} \tag{3.1}$$

where $\beta : R^+ \to R^+$ is a continuous function, not identically equal to zero in R^+.

Remark 3.1. If $c = \infty$ is chosen when $k > 0$, then by the definition of $\Phi(s)$

$$\Phi(s) = -k^{-1}s^{-k}, \quad \forall s \geqslant 0$$

Since $b(t) = \beta(t)A^{-k}(t)$, Eq. (2.4) is replaced by the following relation

$$A(t)F(u(t)) = \left[(A(t_0)F(u_0))^{-k} - k\int_{t_0}^{t} \beta(s)A^{-k}(s)ds\right]^{-1/k}, \quad \forall t \in I \tag{3.2}$$

$$u(t) = u(t, t_0, u_0), \quad t_0 \in R^+, \quad u_0 \in (0, U)$$

The following results stem from Theorems 2.1, 2.3 and 2.4, respectively.

Corollary 3.1. If $k > 0$ and $\beta(t)A^{-k}(t) \in L^1(R^+)$, then the solution $u = 0$ of Eq. (3.1) is uniformly stable. If also $A(t) \leqslant A_1 = \text{const}$ for $t \in R^+$, then this stability is uniform. If also $A(t) \to \infty$ for $t \to \infty$, then the solution $u = 0$ of Eq. (3.1) is equi-asymptotically stable.

Corollary 3.2. If $k < 0$ and

$$A^k(t)\int_0^t \beta(s)A^{-k}(s)ds \to 0 \quad \text{as} \quad t \to \infty \tag{3.3}$$

then the solution $u = 0$ of Eq. (3.1) is eventually stable.

Corollary 3.3. If $k < 0$, $\beta(t) \in L^1(R^+)$ and $A(t) \leqslant A_1 = \text{const}$ in R^+ ($A_1 = \text{const} < \infty$), then the solution $u = 0$ of Eq. (3.1) is eventually uniformly stable.

In addition the following two theorems can be formulated.

Theorem 3.1. Suppose $k > 0$ and two constants $B < \infty$ and $H \geqslant 1$ exist such that

$$1)\ A^k(t)\int_t^\infty \beta(s)A^{-k}(s)ds \leqslant B, \quad \forall t \geqslant 0;$$

$$2)\ A(t_1) \leqslant HA(t_2), \quad \forall t_1 \geqslant 0, \quad \forall t_2 \geqslant t_1.$$

Then the solution $u = 0$ of Eq. (3.1) is uniformly stable.

Proof. Note that Condition 1 only makes sense in the case, when the function $\beta(t)A^{-k}(t)$ belongs to the class $L^1(R^+)$. Hence, by Corollary 3.1 we find that the solution $u = 0$ is stable. We fix ε in the interval $(0, U)$ and determine the constant $\delta = \delta(\varepsilon) < \varepsilon$ from the equality

$$F(\delta)=\left[\left(\frac{H}{F(\varepsilon)}\right)^k+kB\right]^{-1/k}$$

to prove uniform stability.

Taking into account Condition 1 of the theorem we conclude that for any $t_0\in R^+$ and $u_0\in(0,\delta)$ the following inequality holds

$$F^{-k}(u_0)-kA^k(t_0)\int_{t_0}^{t}\beta(s)A^{-k}(s)ds>\left(\frac{H}{F(\varepsilon)}\right)^k,\quad \forall t\geqslant t_0 \tag{3.4}$$

Relation (3.2) can be written in the form

$$A(t)F(u(t))=A(t_0)\left[F^{-k}(u_0)-kA^k(t_0)\int_{t_0}^{t}\beta(s)A^{-k}(s)ds\right]^{-1/k},\quad \forall t\in I$$

Hence, taking into account inequality (3.4) we obtain

$$A(t)F(u(t))<A(t_0)\frac{F(\varepsilon)}{H},\quad \forall t\in I$$

From condition 2 it follows that

$$A(t_0)\leqslant HA(t),\quad \forall t\geqslant t_0$$

and we can conclude that

$$F(u(t))<F(\varepsilon),\quad \forall t\geqslant t_0$$

The theorem is proved.

Theorem 3.2. We will assume that $k<0$, and also that
(1) condition (3.3) is satisfied;
(2) a constant $H\geqslant 1$ exists such that

$$A(t_1)\leqslant HA(t_2),\quad \forall t_1\geqslant 0,\quad \forall t_2\geqslant t_1$$

Then the solution $u=0$ of Eq. (3.1) is eventually uniformly stable.

4. THE CASE OF A PERTURBATION $G(t,u)=\beta(t)g(u)$

We will consider the perturbed equation

$$\dot{u}=a(t)f(u)+\beta(t)g(u) \tag{4.1}$$

where the continuous function $\beta: R^+\to R^+$ belongs to the class $L^1(R^+)$ and $g:(0,U)\to[0,\infty)$ is a continuous function.

We will assume that $a(t)\geqslant 0$ in R^+. Consequently, according to Eq. (1.2) $a(t)$ is a function belonging to the class $L^1(R^+)$.

Theorem 4.1. We will assume that the function $a(t)\beta(t)$ is not identically zero in R^+. Then the zero solution of Eq. (4.1) is uniformly stable, if and only if

$$I(u)=\int_0^y\frac{ds}{f(s)+g(s)}=\infty,\quad \forall u\in(0,U) \tag{4.2}$$

Proof. For condition (4.2) the zero solution of the scalar equation

$$\dot{v} = (a(t) + \beta(t))(f(v) + g(v))$$

is uniformly stable (see, e.g. [1], Corollary 2.1).

From the limit

$$\dot{u} \leqslant (a(t) + \beta(t))(f(u) + g(u))$$

using the comparison method, we conclude the sufficiency.

Suppose condition (4.2) is not satisfied. We will consider the scalar equation

$$\dot{v} = \gamma(t)(f(v) + g(v)); \quad \gamma(t) = \min\{a(t), \beta(t)\}, \quad \forall t \geqslant 0 \tag{4.3}$$

Since the function $\gamma(t)$ is positive and not identically zero in R^+, the solution $u = 0$ of Eq. (4.3) is unstable (see, e.g. [1, Theorem 3.1]). But then from the comparison method we also obtain that the solution $u = 0$ of Eq. (4.1) is unstable. The necessity is proved.

The following theorem is proved similarly.

Theorem 4.2. If in condition (4.2)

$$I(u) < +\infty, \quad \forall u \in (0, U)$$

then the solution $u = 0$ of Eq. (4.1) is eventually uniformly stable.

We will assume that [8]

$$g(u) = f(u)\omega\left(\int_{\bar{u}}^{u} \frac{ds}{f(s)}\right) \tag{4.4}$$

where $\omega : R \to R^+$ is a continuous function.

The following result is a consequence of Theorem 4.1.

Corollary 4.1. If Eq. (4.4) holds and the following integral is diverging

$$\int_{-\infty}^{r} \frac{ds}{1 + \omega(s)} = \infty, \quad \forall r > -\infty$$

then the solution $u = 0$ is uniformly stable.

But if this integral is converging, the solution $u = 0$ is eventually uniformly stable.

5. SCLERONOMOUS MECHANICAL SYSTEMS

We will consider a holonomic mechanical system S with bilateral constraints explicitly independent of time. Let $q^T = (q_1, \ldots, q_N)$ $(N \geqslant 1)$ be the column vector of the independent Lagrange coordinates.

We will denote the kinetic energy of the system S by $T = T(q, \dot{q})$. We will assume that it is the bilinear form of the combined velocities $\dot{q}$, positive definite for any q, belonging to the open connected subset $\Omega \subseteq R^N$, containing the origin of coordinates and also $T \in C^1$. In addition, it is well known [7, 9] that a continuous and non-decreasing scalar function $\alpha : R^+ \to (0, \infty)$ exists such that

$$T(q, \dot{q}) \geqslant \alpha(|q|)|\dot{q}|^2, \quad \forall (q, \dot{q}) \in \Omega \times R^N$$

where $|\cdot|$ is the Euclidian norm in the space R^N.

Suppose certain potential forces act on the system with potential energy

$$\Pi = \Pi(t, q) \quad \Pi \in C^1(R^+ \times \Omega), \quad \Pi(t, 0) \equiv 0$$

and non-potential forces

$$Q = Q(t, q, \dot{q}), \quad Q \in C(R^+ \times \Omega \times R^N)$$

We will assume that each motion of the system is determined for all $t \geqslant t_0$; in other words, the global continuity of the solutions of the corresponding Lagrange equations

$$\frac{d}{dt}(\nabla_{\dot{q}}T(q,\dot{q}))-\nabla_q[T(q,\dot{q})-\Pi(t,q)]=Q(t,q,\dot{q}) \tag{5.1}$$

holds.

Let $V: R^+ \times \Omega \times R^N \to R^+$ be a certain Lyapunov function. We will denote by $\dot{V} = \dot{V}(t, q, \dot{q})$ its derivative, calculated along the motion, i.e.

$$\dot{V}(t,q,\dot{q})=\nabla_{\dot{q}}V(t,q,\dot{q})\ddot{q}+\nabla_q V(t,q,\dot{q})\dot{q}+V_t(t,q,\dot{q})$$

where $\ddot{q}$ is defined by Eq. (5.1) (V_t denotes the partial derivative with respect to t). Note that $\nabla_{\dot{q}}V(t, q, \dot{q})$ and $\nabla_q V(t, q, \dot{q})$ are column vectors in the notation employed and, consequently, $\nabla_{\dot{q}}V(t, q, \dot{q})\ddot{q}$ and $\nabla_q V(t, q, \dot{q})\dot{q}$ are scalar derivatives in R^N.

Suppose $S_x = \{(q, \dot{q}) \in \Omega \times R^N : |x| < r\}$, where $r > 0$ is a sufficiently small real constant and x is identical with $\dot{q}$ in Sections 6 and 7 and with $(q, \dot{q})$ in Section 8.

We will assume that

$$V(t,0,0)=0, \quad \forall t \in R^+ \tag{5.2}$$

$$V(t,q,\dot{q}) \geqslant \gamma_0(|x|), \quad \dot{V}(t,q,\dot{q}) \leqslant g(t, V(t,q,\dot{q})) \tag{5.3}$$

$$\forall(t,q,\dot{q}) \in R^+ \times S_x; \quad \gamma_0 \in K$$

where $g: R^+ \times R^+ \to R$ is a continuous function. We will also assume in some cases that the function $V(t, q, \dot{q})$ allows of an infinitesimal upper limit

$$V(t,q,\dot{q}) \leqslant \gamma_1(|q|+|\dot{q}|), \quad \forall(t,q,\dot{q}) \in R^+ \times S_x \tag{5.4}$$

We will call the corresponding scalar differential equation

$$\dot{u} = g(t, u) \tag{5.5}$$

the *comparison equation.*

We will give the following well-known properties (see [5, 6, 10, 11]) for the convenience of the reader.

Property 1. If the zero solution of Eq. (5.5) is stable (asymptotically stable), then the zero equilibrium position of system S is stable (asymptotically stable) with respect to x.

Property 2. If the zero solution of Eq. (5.5) is uniformly stable and V satisfies inequality (5.4), then the zero equilibrium position of system S is uniformly stable with respect to x.

Similar conclusions also hold for the eventual stability.

Remark 5.1. We will assume the uniqueness of all the solutions of the system of Lagrange equations (5.1) and that the following conditions are satisfied

$$\nabla_q \Pi(t,0) \equiv 0, \quad Q(t,0,0) \equiv 0, \quad \forall t \in R^+$$

This means that $q(t) \equiv 0$, $\dot{q}(t) \equiv 0$, i.e., the zero equilibrium position of system S is the solution of the system of Lagrange equations. Consequently, in this case the eventual stability implies the stability [6].

These assumptions are omitted for the possible extension of the results obtained and only the conditions of eventual stability are given in the corresponding results.

6. THE BASIC DIFFERENTIAL INEQUALITY

We will assume $S_{\dot{q}} = \{\dot{q} \in R^N : |\dot{q}| < r\}$. In this section and in Sections 7 and 8 we will assume that the following conditions are satisfied

$$\Pi(t,q) \geqslant -p(\sigma(t)\tau(|q|)), \quad \forall(t,q) \in R^+ \times \Omega \tag{6.1}$$

$$T(q,\dot{q}) \geqslant [\tau'(|q|)|\dot{q}|]^2, \quad \forall(q,\dot{q}) \in \Omega \times R^N \tag{6.2}$$

$$Q^T(t,q,\dot q)\dot q + \partial\Pi/\partial t \leqslant a(t)f(V) - 2\dot\sigma(t)\tau(|q|)p'(\sigma(t)\tau(|q|)) \tag{6.3}$$

$$\forall(t,q,\dot q)\in R^+\times\Omega\times S_{\dot q}$$

Here

$$V(t,q,\dot q) = E(t,q,\dot q) + 2p(\sigma(t)\tau(|q|)) \tag{6.4}$$

is the Lyapunov function, which is the perturbed total energy

$$E(t,q,\dot q) = T(q,\dot q) + \Pi(t,q)$$

We will also assume that

(1) the function $p = p(s)$ belongs to the class $C^1(R^+)$, $p(0) = 0$ and its derivative $p'(s)$ is an increasing and strictly positive function for $s > 0$;

(2) the function $\sigma = \sigma(t)$ belongs to the class $C^1(R^+)$ and is strictly positive in R^+;

(3) the function $\tau = \tau(s)$ belongs to the class $C^1(R^+)$, $\tau(0) = 0$ and a constant $\varepsilon > 0$ exists such that

$$\tau'(s) \geqslant \varepsilon, \quad \forall s\in R^+$$

(4) the functions $a = a(t)$ and $f = f(s)$ are defined and continuous in R^+, $f(s) > 0$ when $s > 0$ and

$$\int_0^u \frac{ds}{f(s)} = \infty, \quad \forall u > 0$$

Remark 6.1. Since $T(q, \dot q) \geqslant [\varepsilon|\dot q|]^2$ in $\Omega\times R^N$, conditions (6.1) and (6.2) can be replaced by the following

$$\Pi(t,q) \geqslant -p(\sigma(t)|q|), \quad \forall(t,q)\in R^+\times\Omega$$
$$T(q,\dot q) \geqslant [\varepsilon|\dot q|]^2, \quad \forall(q,\dot q)\in\Omega\times R^N$$

We will obtain the limit for the derivative $\dot V(t, q, \dot q)$ in order to apply the comparison method. We have

$$\dot V(t,q,\dot q) \leqslant a(t)f(V) + 2\sigma(t)p'(\sigma(t)\tau(|q|))\nabla_q\tau(|q|)\dot q \tag{6.5}$$

$$\forall(t,q,\dot q)\in R^+\times\Omega\times R^N$$

On the other hand, $|\nabla_q\tau(|q|)| = \tau'(|q|)$ on Ω, and thus it can be seen that

$$\nabla_q\tau(|q|)\dot q \leqslant \tau'(|q|)|\dot q|, \quad \forall(q,\dot q)\in\Omega\times R^N$$

Consequently, taking into account condition (6.2) we will have

$$\nabla_q\tau(|q|)\dot q \leqslant \sqrt{V(t,q,\dot q)}, \quad \forall(t,q,\dot q)\in R^+\times\Omega\times R^N \tag{6.6}$$

From condition (6.1) we find

$$\sigma(t)\tau(|q|) \leqslant p^{-1}(V(t,q,\dot q)), \quad \forall(t,q,\dot q)\in R^+\times\Omega\times R^N$$

and hence, taking into account that the function $p'(s)$ is an increasing function, we obtain

$$p'(\sigma(t)\tau(|q|)) \leqslant (p'\circ p^{-1})(V(t,q,\dot q)), \quad \forall(t,q,\dot q)\in R^+\times\Omega\times R^N \tag{6.7}$$

From inequality (6.5), taking into account inequalities (6.3), (6.6) and (6.7), we finally obtain the following differential inequality

$$\dot V \leqslant a(t)f(V) + 2\sigma(t)\sqrt{V}(p'\circ p^{-1})(V), \quad \forall(t,q,\dot q)\in R^+\times\Omega\times S_{\dot q} \tag{6.8}$$

Consequently, taking into account that

$$V(t,q,\dot{q})\geq\varepsilon^2\,|\dot{q}|^2, \quad \forall(t,q,\dot{q})\in R^+\times\Omega\times S_{\dot{q}}$$

we conclude that both conditions (5.3) are satisfied for $x = \dot{q}$ and the corresponding comparison equation has the form $\dot{u} = g(t, u)$, where $g : R^+ \times R^+ \to R$ is any continuous function satisfying the condition

$$a(t)f(s)+2\sigma(t)\sqrt{s}(p'\circ p^{-1})(s)\leq g(t,s), \quad \forall(t,s)\in R^+\times R^+ \tag{6.9}$$

7. STABILITY WITH RESPECT TO A PART OF THE VARIABLES WITH RESPECT TO $\dot{q}$

We will define the function $c : R^+ \to R^+$ by the equality

$$c(t)=\max\{a(t),2\sigma(t)\}, \quad \forall t\in R^+$$

Theorem 7.1. If $c(t) \in L^1(R^+)$ and

$$J(u)=\int_0^u \frac{ds}{f(s)+\sqrt{s}(p'\circ p^{-1})(s)}=\infty, \quad \forall u>0 \tag{7.1}$$

then the equilibrium position $\dot{q} = q = 0$ of system S is stable with respect to $\dot{q}$. If also $V(t, q, \dot{q})$ satisfies condition (5.4), then the stability with respect to $\dot{q}$ is uniform.

Proof. The continuous function $g(t, s)$, defined by the relation

$$g(t,s)=c(t)[f(s)+\sqrt{s}(p'\circ p^{-1})(s)] \tag{7.2}$$

satisfies inequality (6.9).

Applying Corollary 2.1 from [1], it is also possible to show that the zero solution of the comparison method $\dot{u} = g(t, u)$ is uniformly stable. Properties 1 and 2 complete the proof.

Theorem 7.2. If $c(t) \in L^1(R^+)$ and the integral $J(u)$ (7.1) converges, the equilibrium position $\dot{q} = q = 0$ is eventually stable with respect to $\dot{q}$. If inequality (6.4) holds, this stability is uniform.

The proof, as in the previous case, is derived from the fact that the solution $u = 0$ of the equation $\dot{u} = g(t, u)$, where $g(t, s)$ is defined by relation (7.2), is uniformly stable [1, Corollary 3.1]. We will assume

$$r(u)=\int_{\bar{u}}^u \frac{ds}{f(s)}$$

and suppose that a continuous function $h : R \to R^+$ exists such that

$$\sqrt{u}(p'\circ p^{-1})(u)\leq f(u)h(r(u)), \quad \forall u\in R^+ \tag{7.3}$$

where $\bar{u} > 0$ is an appropriately chosen real constant. Consequently

$$a(t)f(u)+2\sigma(t)\sqrt{u}(p'\circ p^{-1})(u)\leq c(t)f(u)[1+h(r(u))], \quad \forall u\in R^+ \tag{7.4}$$

Since

$$J_1(r)=\int_0^u \frac{ds}{f(s)[1+(h\circ r)(s)]}=\int_{-\infty}^{r(u)} \frac{ds}{1+h(s)}, \quad \forall u\in R^+$$

then the following results can be easily obtained.

Corollary 7.1. Suppose $c(t) \in L^1(R^+)$. If condition (7.3) holds and $J_1(r) = \infty$, $\forall r > -\infty$, then all the conditions of Theorem 7.1 are satisfied.

Corollary 7.2. We will assume that $c(t) \in L^1(R^+)$ and $J_1(r) < \infty$ for $r > -\infty$. Then all the conditions of Theorem 7.2 are satisfied.

Remark 7.1. We will assume that $h(s) \leq H$, $\forall s \in R$, where $H < \infty$ is a real constant. Then the right-hand side of condition (7.4) can be replaced by $\tilde{a}(t)f(u)$, where $\tilde{a}(t) = a(t) + 2H\sigma(t)$.

It follows that the zero solution of the comparison equation $\dot{u} = g(t, u)$ is stable when (and only when) the function $\tilde{a}(t)$ belongs to the class $L^1(R^+)$. This means, in particular, that the equilibrium position $\dot{q} = q = 0$ of system S will be stable (or uniformly stable) with respect to $\dot{q}$ also when $c(t) \notin L^1(R^+)$.

Moreover, if

$$\int_0^t \tilde{a}(s)ds \to -\infty \quad \text{as} \quad t \to \infty$$

then the zero solution of Eq. (5.5) is equi-asymptotically stable. [1].

Thus, it can be proved that the zero equilibrium position of system S is equi-asymptotically stable with respect to $\dot{q}$.

Remark 7.2. We will assume that $p(s) = s^n$ $(n \geq 1)$ and $f(s) = s$. Then inequality (7.3) holds when

$$h(s) = n\exp\left(\frac{n-2}{2n}s\right)$$

Consequently, when $1 \leq n < 2$ we can apply Corollary 7.1, and when $n \geq 2$ we can apply Corollary 7.2. The case $n = 2$ can also be included in the previous Remark 7.1.

We will consider the case when $a(t) \leq 0$, $\forall t \in R^+$.

Theorem 7.3. We will assume that $a(t) \leq 0$, $\forall t \in R^+$, the function $\sigma(t)$ belongs to the class $L^1(R^+)$ and

$$J_2(r) = \int_0^r \frac{ds}{\sqrt{p(s)}} = \infty, \quad \forall r > 0$$

Then the zero equilibrium position of system S is stable with respect to $\dot{q}$. The stability will be uniform, if Eq. (5.4) is satisfied.

Proof. Assuming $r(s) = p^{-1}(s)$, it is obvious that

$$\int_0^u \frac{ds}{\sqrt{s}(p' \circ p^{-1})(s)} = \int_0^{r(u)} \frac{ds}{\sqrt{p(s)}}$$

Consequently, since $r(s) \to 0$ for $s \to 0$, the conditions of the theorem will ensure uniform stability of the zero solution of the comparison equation.

$$\dot{u} = 2\sigma(t)\sqrt{u}(p' \circ p^{-1})(u)$$

Properties 1 and 2 complete the proof.

The following theorem can be proved similarly.

Theorem 7.4. We will assume that $a(t) \leq 0$, $\forall t \in R^+$ and $\sigma(t) \in L^1(R^+)$, $J_2(r) < \infty$, $\forall r > 0$. Then the equilibrium position $\dot{q} = q = 0$ is eventually stable with respect to $\dot{q}$. If Eq. (5.4) is satisfied, this stability will be uniform.

We will consider a special case, for which $h(s) = s^k$ (k is a real constant) in Eq. (7.3). Consequently, inequality (6.9) holds, where

$$g(t, u) = a(t)f(u) + 2\sigma(t)f(u)\exp(r(u))$$

We will also assume that the function $a(t)$ satisfies Eq. (1.2), but the function $c(t)$ does not necessarily belong to the class $L^1(R^+)$.

Theorem 7.5. We will assume that $k > 0$. Then, if $\sigma(t)A^{-k}(t) \in L^1(R^+)$, the zero equilibrium position of system S is uniform with respect to $\dot{q}$.

Proof. According to Corollary 3.1, the conditions of the theorem ensure that the zero solution of the comparison equation

$$\dot{u} = g(t,u)$$

is stable. Property 1 completes the proof.

The following theorems are proved similarly, on the basis of Theorem 3.1.

Theorem 7.6. We will assume that Eq. (5.4) is satisfied, $k > 0$ and also
(1) a constant $B < \infty$ exists such that

$$A^k(t)\int_t^\infty \sigma(s)A^{-k}(s)ds \leq B, \quad \forall t \in R^+$$

(2) a constant $H \geq 1$ exists such that

$$A(t_1) \leq HA(t_2), \quad \forall t_1 \geq 0, \ \forall t_2 \geq t_1$$

Then the zero equilibrium position of system S is uniformly stable with respect to $\dot{q}$.

Theorem 7.7. We will assume that $k < 0$ and

$$A^k(t)\int_0^t \sigma(s)A^{-k}(s)ds \to 0, \quad t \to \infty$$

Then the zero equilibrium position is eventually stable with respect to $\dot{q}$.
If additionally Eq. (5.4) holds and condition 2 of Theorem 7.6 is satisfied, then the zero equilibrium position is eventually uniformly stable with respect to $\dot{q}$.

8. STABILITY WITH RESPECT TO $(q, \dot{q})$

Assuming $S_q = \{q \in \Omega : |q| < r\}$, we will suppose that

$$\Pi(t,q) \geq -p(\sigma(t)\tau(|q|)) + \varphi(|q|), \quad \forall (t,q) \in R^+ \times S_q \tag{8.1}$$

and also that conditions (6.2) and (6.3) are satisfied for all $(t, q, \dot{q}) \in R^+ \times S_q \times S_{\dot{q}}$ with the change and addition that $\tau = \tau(s)$ is a non-negative and non-decreasing function belonging to the class $C^1(R^+)$ and $\varphi = \varphi(s)$ belongs to the class K.

Remark 8.1. Inequality (5.1) and condition (8.1) ensure that the inequality

$$V(t,q,\dot{q}) \geq a(r)|\dot{q}|^2 + \varphi(|q|), \quad \forall (t,q,\dot{q}) \in R^+ \times S_q \times S_{\dot{q}}$$

holds and, consequently, Eq (5.3) holds for $x = (q, \dot{q})$. Moreover, as in Section 6, it is possible to obtain a differential inequality, which differs from inequality (6.8) only in the fact that it holds for all $(t, q, \dot{q}) \in R^+ \times S_q \times S_{\dot{q}}$. Hence, denoting the function satisfying condition (6.9) by $g(t, s)$, it is clear that Eq. (5.4) also holds.

The following stability criteria are essentially similar to the stability criteria with respect to a part of the variables in Section 7. We will therefore confine ourselves to formulating two theorems, which, in turn, are obtained from Theorems 3.1 and 3.2. In these theorems

$$J(u) = \int_0^u \frac{ds}{f(s) + \sqrt{s}(p' \circ p^{-1})(s)}$$

Theorem 8.1. If the function $c(t)$ belongs to the class $L^1(R^+)$ and $J(u) = \infty$, $\forall u > 0$, the zero equilibrium position of system S is stable. Moreover, the stability will be uniform if Eq. (5.4) is satisfied.

Theorem 8.2. If the function $c(t) \in L^1(R^+)$ and $J(u) < \infty$, $\forall u > 0$, the zero equilibrium position of system S is eventually stable and consequently, for the condition (5.4) it is eventually uniformly stable.

9. EXAMPLES

Example 1. Consider a heavy particle P of mass m, moving along the parabola

$$y = x^2/2p, \quad z = 0 \ (p = \text{const} > 0)$$

in the system of coordinates $Oxyz$. The system of coordinates $Oxyz$ rotates with an angular velocity $\boldsymbol{\omega}(t) = \omega(t)\mathbf{j}$ (the unit vectors of the x, y, z axes are denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively) around the vertical y axis, relative to the coordinate system connected to the Earth, which we assume to be inertial. Then

$$\sigma^2(t) = \omega^2(t) - g/p > 0, \quad \forall t \in R^+ \tag{9.1}$$

Taking the abscissa x as a Lagrange coordinate, we will have

$$T(x,\dot{x}) = \frac{1}{2}m\left(1 + \frac{x^2}{p^2}\right)\dot{x}^2, \quad \Pi(t,x) = -\frac{1}{2}m\sigma^2(t)x^2 \tag{9.2}$$

and $Q(t, x, \dot{x}) \equiv 0$.

Note that the potential energy is negative near $x = 0$ for fixed $t \in R^+$. Nevertheless, applying Theorem 7.1, we obtain that if the function $\sigma(t)$ (9.1) is bounded in R^+ and

$$\int_0^\infty \sigma(t)dt < \infty$$

then the equilibrium position $\dot{x} = x = 0$ is uniformly stable with respect to $\dot{x}$.

Example 2. We will consider Example 1, assuming in addition that an elastic force $\mathbf{f} = -k\mathbf{OP}$ is acting on the particle P, where the time-dependent stiffness $k = k(t)$ satisfies the inequality

$$k_* \leqslant k(t) < m\sigma^2(t), \quad \forall t \in R^+, \quad k_* = \text{const} > 0$$

Applying Theorem 8.1, it can be shown that if the function

$$c(t) = \max\left\{2\sqrt{\sigma^2(t) - k(t)/m}, \ \frac{2w(t)\dot{w}(t) - \dot{k}(t)/m}{\sqrt{\sigma^2(k) - k(t)/m}}, \ \frac{\dot{k}(t)}{k(t)}\right\}$$

belongs to the class $L^1(R^+)$, the equilibrium position $\dot{x} = x = 0$ is stable.

Example 3. We will assume $\mathbf{w}(t) \equiv 0$ in Example 1. Additionally we will assume that a time-dependent force $\mathbf{f} = \phi(t)\mathbf{i}$, where $\phi(t) > 0$, $\forall t \in R^+$, acts on the (heavy) particle P. The kinetic energy is defined by Eq. (9.1) and the generalized potential energy will have the form

$$\Pi(t,x) = -\phi(t)x + \frac{1}{2}\frac{mg}{p}x^2$$

and $Q(t, x, \dot{x}) \equiv 0$. Applying Theorem 8.2, it can be shown that if the function $\phi(t)$ is bounded in R^+ and

$$\int_0^\infty \phi(t)dt < \infty$$

then the equilibrium position $\dot{x} = x = 0$ is eventually uniformly stable. Note that in this case $\dot{x} = x = 0$ is not a solution of the Lagrange equation. Hence, in the given example eventual stability does not ensure stability.

Example 4. Consider a heavy particle P of mass m, moving along the curve

$$y = \frac{4}{7}|x|^{7/4}, \quad z = 0$$

in the system of coordinates $Oxyz$, connected to the Earth; the y axis is vertical. We will assume that a time-dependent force $\mathbf{f} = m\phi(t)\mathbf{j}$ is acting on the particle, where

$$\phi(t) > g, \quad \dot{\phi}(t) \leqslant 0, \quad \forall t \in R^+$$

We have

$$T = \frac{1}{2}m(1 + |x|^{3/2})\dot{x}^2, \quad \Pi(t,x) = -\frac{4}{7}m(\phi(t) - g)|x|^{7/4}, \quad Q(t,x,\dot{x}) \equiv 0$$

Applying Theorem 7.4 it can be shown that if

$$\int_0^{\infty} (\phi(t)-g)^{4/7}\, dt < \infty$$

then the equilibrium position $\dot{x} = x = 0$ is eventually uniformly stable with respect to $\dot{x}$. In this case $\dot{x} = x = 0$ is the solution of the Lagrange equation

$$(1+|x|^{3/2})\ddot{x} + \frac{3}{4}\frac{x}{|x|^{1/2}}\dot{x}^2 + (g-\phi(t))\frac{x}{|x|^{1/4}} = 0$$

but the sufficient conditions for the uniqueness of the zero solution are not satisfied.

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